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Invariant Subspaces and Extremum Problems in Spaces of Vector-Valued Functions

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In this article we obtain, for $1 \leqslant p \leqslant \infty$, a characterization of the invariant subspaces of spaces of vector-valued L^p functions defined on the unit circle—i.e., of those subspaces invariant under multiplication by e^{ix}. This result is then applied to extend, to the corresponding Hardy classes of vector-valued functions, the known characterizations of the extreme points of the unit ball in the scalar Hardy classes H^1 and H^∞. Finally, it is shown that the characterization of the closure of the set of extreme points of the unit ball in H^1 changes significantly when we pass from the scalar to the vector case.

Let $\mathscr{H}$ be a complex, separable Hilbert space, and for $1 \leqslant p \leqslant \infty$, let $L^p_{\mathscr{H}}$ denote the Banach space of weakly measurable functions F defined on the unit circle to $\mathscr{H}$ for which the norm

$$\| F \|_p = \left\{ \int \| F(e^{ix}) \|^p \, d\sigma \right\}^{1/p}, \qquad p < \infty,$$

$$\| F \|_\infty = \operatorname{ess\,sup} \| F(e^{ix}) \|$$

is finite. (Here $\| \cdot \|_p$ denotes the norm in $L^p_{\mathscr{H}}$ and $\| \cdot \|$ that in $\mathscr{H}$, while σ is the normalized Lebesgue measure on the circle.) Throughout this paper, the interaction between the elements of any Banach space and those of its dual, as well as the inner product in $\mathscr{H}$, will be denoted by $\langle \cdot, \cdot \rangle$.

Let χ denote the scalar function defined on the unit circle by $\chi(e^{ix}) = e^{ix}$. Finite linear combinations of nonnegative powers of χ will be referred to as *trigonometric polynomials*. If F belongs to $L^1_{\mathscr{H}}$, the largest of the spaces $L^p_{\mathscr{H}}$, F has a Fourier series

$$F(e^{ix}) \sim \sum_{k=-\infty}^{\infty} \varphi_k \chi^k,$$

where the Fourier coefficients φ_k are elements of $\mathscr{H}$, [4, p. 53]. We denote

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by $H^p_{\mathscr{H}}$ the subspace of $L^p_{\mathscr{H}}$ consisting of those functions whose Fourier coefficients vanish for negative indices, and by $K^p_{\mathscr{H}}$ the subspace consisting of functions whose Fourier coefficients vanish for positive indices. We will often refer to elements of $H^p_{\mathscr{H}}$ as analytic functions, and those of $K^p_{\mathscr{H}}$ as conjugate analytic functions. If $\mathscr{H}$ is one dimensional, we denote $L^p_{\mathscr{H}}$ by L^p and $H^p_{\mathscr{H}}$ by H^p. An equivalent definition of $H^p_{\mathscr{H}}$, if the dimension of $\mathscr{H}$ is greater than or equal to 2, is then the following: A function F in $L^p_{\mathscr{H}}$ belongs to $H^p_{\mathscr{H}}$ if, and only if, $\langle F, e\rangle \in H^p$ for all $e \in \mathscr{H}$ [4, p. 55]. Similarly, a function $F \in L^p_{\mathscr{H}}$ belongs to $K^p_{\mathscr{H}}$ if, and only if, $\langle e, F\rangle \in H^p$ for all $e \in \mathscr{H}$. A thorough discussion of these spaces may be found in [4].

A closed subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ is called *invariant* if $\chi F \in \mathscr{M}$ for all $F \in \mathscr{M}$. $\mathscr{M}$ is called *conjugate invariant* if $\chi^{-1}F \in \mathscr{M}$ for all $F \in \mathscr{M}$. $\mathscr{M}$ is called *doubly invariant* if it is invariant and conjugate invariant. $\mathscr{M}$ is called *simply invariant* if it is invariant but not conjugate invariant. $\mathscr{M}$ is called *simply conjugate invariant* if it is conjugate invariant but not invariant.

In recent years, considerable work has been directed toward the generalization of two classic results on invariant subspaces due to Wiener and Beurling. The characterization of doubly invariant subspaces of L^2 (Wiener's theorem) was extended to $L^2_{\mathscr{H}}$ by Srinivasan [8], and then to $L^p_{\mathscr{H}}$ by Hasumi and Srinivasan [3]. The characterization of simply invariant subspaces of H^2 (Beurling's theorem), was extended to L^2 by Helson and Lowdenslager [5], then to L^p by Srinivasan and Wang [9]. Moving toward vector functions, Lax extended Beurling's theorem to $H^2_{\mathscr{H}}$ [7], and then Helson gave the formulation for $L^2_{\mathscr{H}}$. However, no description of the simply invariant subspaces of $L^p_{\mathscr{H}}$, for $1 \leqslant p \leqslant \infty$, seems to exist in the literature.

In Section 1 we give such a description. The proof of Theorem 1, for the case $p < 2$, is merely sketched, since it requires only arguments previously employed by Hasumi, Helson, Srinivasan, and Wang. However in the case $p > 2$, certain arguments not previously used by these authors are necessary, although our proof is quite obviously in the spirit of their work.

In Section 2 we apply Theorem 1 to solve certain extremum problems in $H^1_{\mathscr{H}}$ and $H^\infty_{\mathscr{H}}$. In [1], de Leeuw and Rudin investigated extremum problems in the space H^1; in particular they gave a description of the set of extreme points of the unit ball of H^1, and of the norm closure of this set. A description of the extreme points of the unit ball of H^∞ is found in [6]. Here we consider the vector analogs of these, and related, results. Using a factorization of $H^1_{\mathscr{H}}$ elements which is provided by Theorem 1, and a definition of *outer function* which was originally suggested by Helson for elements of $H^p_{\mathscr{H}}$, with $p \geqslant 2$, we arrive at the precise analogs of the descriptions of the set of extreme points of the unit ball in the corresponding spaces of scalar functions. There are other means of deriving these results, but I believe that the proofs given here are about as short as possible. This is particularly true of Theorems 3 and 4, which due to our factorization become immediate corollaries of known results for scalar functions. Finally, we show that when we consider the closure of the set of extreme points in $H^1_{\mathscr{H}}$, the situation changes radically as we pass from the scalar to the vector case.

1. Invariant Subspaces

We will call P a *measurable range function* if P is a function defined a.e. on the unit circle to the space of orthogonal projections on $\mathscr{H}$ wich is weakly measurable. We shall denote by $\hat{P}$ the operator on $L^p_{\mathscr{H}}$ defined by $(\hat{P}F)(e^{ix}) = P(e^{ix})F(e^{ix})$ a.e. Moreover, given an invariant subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$, we shall denote by $\mathscr{M}_\infty$ the subspace $\bigcap_{k=0}^{\infty} \chi^k\mathscr{M}$ of $\mathscr{M}$, and call $\mathscr{M}_\infty$ the *remote past* of $\mathscr{M}$. If $\mathscr{M}_\infty = \{0\}$, we shall say that $\mathscr{M}$ has *trivial remote past*. Finally, if $e \in \mathscr{H}$, we denote by $\mathbf{e}$ that element of $L^p_{\mathscr{H}}$ which is constantly equal to e.

THEOREM 1. *Let $\mathscr{M}$ be a closed, simply invariant subspace of $L^p_{\mathscr{H}}$, $1 \leqslant p \leqslant \infty$ (weak $*$ closed if $p = \infty$). Then there exists a Hilbert space $\mathscr{H}_1$ and a weakly measurable operator-valued function U defined on the circle, where, for almost all e^{ix}, $U(e^{ix})$ is an isometry of $\mathscr{H}_1$ into $\mathscr{H}$. If we denote by $\hat{U}H^p_{\mathscr{H}_1}$ the set of all $G \in L^p_{\mathscr{H}}$ of the form $G(e^{ix}) = U(e^{ix})F(e^{ix})$, for $F \in H^p_{\mathscr{H}_1}$, then*

$$\mathscr{M} = \hat{U}H^p_{\mathscr{H}_1} \oplus \hat{P}L^p_{\mathscr{H}}$$

for some measurable range function P.

Proof. First suppose that $1 \leqslant p < 2$. $\mathscr{M}_\infty$ is a closed, doubly invariant subspace of $L^p_{\mathscr{H}}$, and hence by [3, p. 531], $\mathscr{M}_\infty = \hat{P}L^p_{\mathscr{H}}$, for some measurable range function P. Let $\mathscr{M}' = (I - P)^\wedge L^p_{\mathscr{H}} \cap \mathscr{M}$ (where I is, of course, the identity operator in $\mathscr{H}$). Then $\mathscr{M}'$ is a closed, simply invariant subspace of $L^p_{\mathscr{H}}$ which has trivial remote past, and $\mathscr{M} = \mathscr{M}' \oplus \mathscr{M}_\infty$. Note that every function in $\mathscr{M}'$ is a.e. orthogonal on the circle to the range of P.

Now let $\mathscr{N} = \mathscr{M}' \cap L^2_{\mathscr{H}}$. Then $\mathscr{N}$ is a closed, invariant subspace of $L^2_{\mathscr{H}}$, and the exact analog of the argument given for the scalar L^p case in [4, p. 26] shows that $\mathscr{N}$ is a simply invariant subspace of $L^2_{\mathscr{H}}$ having trivial remote past, which is dense in $\mathscr{M}'$. Thus [4, p. 61], $\mathscr{N} = \hat{U}H^2_{\mathscr{H}_1}$, for some Hilbert space $\mathscr{H}_1$, where $\hat{U}$ is as described in the statement of the theorem. It then follows readily from the facts that $H^2_{\mathscr{H}_1}$ is dense in $H^p_{\mathscr{H}_1}$, and that $\hat{U}$ is an isometry with respect to the vectorial L^p norm, that $\mathscr{M}' = UH^p_{\mathscr{H}_1}$. This concludes the proof for $1 \leqslant p < 2$.

Next suppose that $2 < p \leqslant \infty$, and let q be defined by $(1/p) + (1/q) = 1$, $p < \infty$, and $q = 1$ if $p = \infty$. It is known [2, p. 282] that the dual space of

$L^q_{\mathscr{H}}$ is $L^p_{\mathscr{H}}$, where the interaction between elements $F \in L^q_{\mathscr{H}}$ and $G \in L^p_{\mathscr{H}}$ is given by $\langle F, G\rangle = \int \langle F(e^{ix}), G(e^{ix})\rangle\, d\sigma$. Let ${}^0\mathscr{M} = \{F \in L^q_{\mathscr{H}} : \langle F, G\rangle = 0$, all $G \in \mathscr{M}\}$. Then ${}^0\mathscr{M}$ is a closed, simply conjugate invariant subspace of $L^q_{\mathscr{H}}$, and by the analog of what we have already established, applied to simply conjugate invariant subspaces, ${}^0\mathscr{M} = \hat{U}K^q_{\mathscr{H}_1} \oplus \hat{P}_0 L^q_{\mathscr{H}}$, for some measurable range function P_0 and some Hilbert space $\mathscr{H}_1$, where $\hat{U}$ is, of course, an isometry of $K^q_{\mathscr{H}_1}$ into $L^q_{\mathscr{H}}$ derived from an a.e. pointwise isometric mapping $U(e^{ix})$ of $\mathscr{H}_1$ into $\mathscr{H}$.

Let $\{\epsilon_j : j = 1, 2,\ldots\}$ be an orthonormal basis for $\mathscr{H}_1$ and let $P_1(e^{ix})$ be equal to the orthogonal projection of $\mathscr{H}$ onto the closed linear span of $\{U(e^{ix})\,\epsilon_j$: $j = 1, 2,\ldots\}$. Then P_1 is a measurable range function and, again by the conjugate analytic analog of what we have proven above, the range of $P_1(e^{ix})$ is a.e. orthogonal in $\mathscr{H}$ to the range of $P_0(e^{ix})$. Hence $P_2 = P_0 + P_1$ is a measurable range function.

Now let P be the measurable range function defined by $P = I - P_2$. Since the annihilator of ${}^0\mathscr{M}$ in $L^p_{\mathscr{H}}$ is precisely $\mathscr{M}$, it is clear that $\hat{P}L^p_{\mathscr{H}} \subseteq \mathscr{M}$. Also, using the facts that any element of $\chi H^p_{\mathscr{H}_1}$ annihilates $K^q_{\mathscr{H}_1}$, that $\langle F, \chi G\rangle = \langle \hat{U}F, (\chi U)^\wedge G\rangle$ for $F \in K^q_{\mathscr{H}_1}$ and $G \in H^p_{\mathscr{H}_1}$, and that for such G, $(\chi U)^\wedge G$ is pointwise orthogonal on the circle to any element of $\hat{P}_0 L^q_{\mathscr{H}}$, it follows that $(\chi U)^\wedge H^p_{\mathscr{H}_1} \subseteq \mathscr{M}$. Hence $(\chi U)^\wedge H^p_{\mathscr{H}_1} \oplus \hat{P}L^p_{\mathscr{H}} \subseteq \mathscr{M}$.

We wish to establish the reverse inclusion. Let $\{e_n : n = 1, 2,\ldots\}$ be an orthonormal basis for $\mathscr{H}$. It is a consequence of the proof of Theorem 3 in [8, p. 704] that the closed linear span of $\{P_0(e^{ix})e_n : n = 1, 2,\ldots\}$ is a.e. equal to $P_0(e^{ix})\,\mathscr{H}$. Since $\hat{P}_0 L^q_{\mathscr{H}}$ is doubly invariant, it follows that if $G \in \mathscr{M}$, then for all $n \geqslant 1$, and $k = 0, \pm 1, \pm 2,\ldots$, we have $0 = \langle \chi^k \hat{P}_0 \mathbf{e}_n, G\rangle = \int \chi^k \langle P_0(e^{ix})\, e_n, G(e^{ix})\rangle\, d\sigma$, so that $G(e^{ix})$ is a.e. orthogonal to $P_0(e^{ix})\,\mathscr{H}$.

Hence if G is any element of $\mathscr{M}$, G can be written uniquely as $G = G_1 + G_2$, where $G_2 = \hat{P}G \in \hat{P}L^p_{\mathscr{H}}$, and $G_1(e^{ix}) = (\hat{P}_1 G)(e^{ix}) \in P_1(e^{ix})\,\mathscr{H}$ a.e. By the definition of P_1, we have $G_1(e^{ix}) = \sum_{j\geqslant 1} g_j(e^{ix})\, U(e^{ix})\,\epsilon_j$, for certain scalar functions g_j. And as $g_j(e^{ix}) = \langle G_1(e^{ix}), U(e^{ix})\,\epsilon_j\rangle$, each g_j is a measurable function, and, in fact, an element of L^p. Moreover, we know that each element of $\hat{U}K^p_{\mathscr{H}_1}$ annihilates G_1, and for all $k \geqslant 0$ and $j = 1, 2,\ldots$, $\chi^{-k}\hat{U}\boldsymbol{\epsilon}_j \in \hat{U}K^q_{\mathscr{H}_1}$. Thus the same type of argument used in the preceding paragraph shows that, for each j, the Fourier coefficients of g_j that are indexed by the nonpositive integers all vanish, so that each $g_j \in \chi H^p$ and hence $G_1 \in (\chi U)^\wedge H^p_{\mathscr{H}_1}$. We have thus proved that $\mathscr{M} = (\chi U)^\wedge H^p_{\mathscr{H}_1} \oplus \hat{P}L^p_{\mathscr{H}}$.

2. Outer Functions and Extremum Problems

The following definition is due to Helson for elements of the spaces $H^p_{\mathscr{H}}$ with $p \geqslant 2$.

DEFINITION 1. An element F of $H^1_{\mathscr{H}}$ is an *outer function* if the only scalar inner functions q such that $\bar{q}F \in H^1_{\mathscr{H}}$ are constant functions.

DEFINITION 2. An element E of $L^1_{\mathscr{H}}$ will be called an *analytic unitary function* if $E \in H^1_{\mathscr{H}}$ and $\| E(e^{ix})\| = 1$ a.e.

Now suppose that F is any nonzero element of $H^1_{\mathscr{H}}$. Let P be the measurable range function such that $P(e^{ix})\mathscr{H}$ is a.e. the one-dimensional subspace of $\mathscr{H}$ containing $F(e^{ix})$. The set $\mathscr{M}$ of all elements $G \in H^1_{\mathscr{H}}$ such that $G(e^{ix}) \in P(e^{ix})\mathscr{H}$ a.e. is clearly a closed, invariant subspace of $H^1_{\mathscr{H}}$ which (since it lies in $H^1_{\mathscr{H}}$), must be simply invariant and have trivial remote past. Thus, by Theorem 1, there exists a Hilbert space $\mathscr{H}_1$ and an isometry $\hat{U}$ such that $\mathscr{M} = \hat{U}H^1_{\mathscr{H}_1}$. It is clear that $\mathscr{H}_1$ is one-dimensional, so that if $\epsilon \in \mathscr{H}_1$ and $\| \epsilon \| = 1$, and if we set $E = \hat{U}\epsilon$, then every element of $\mathscr{M}$ (and thus, in particular, F) is of the form of fE for some $f \in H^1$. Then E is an analytic unitary function and is easily seen to be outer.

DEFINITION 3. If $F \in H^1_{\mathscr{H}}$, $F \neq 0$, an element $E \in H^1_{\mathscr{H}}$, related to F as in the preceding paragraph, is called an *outer analytic unitary function in the direction of F.*

It is easy to see that if E is any analytic unitary function, and if E is outer, E is an outer analytic unitary function in the direction of any $H^1_{\mathscr{H}}$ element which is pointwise a scalar multiple of E.

THEOREM 2. *Let $F \in H^1_{\mathscr{H}}$. Then F is an extreme point of the unit ball in $H^1_{\mathscr{H}}$ if, and only if, F is an outer function of norm one.*

Proof. First assume that F is outer and of norm one. Suppose that $F = \frac{1}{2}(F_1 + F_2)$, where $F_j \in H^1_{\mathscr{H}}$ and $\|F_j\|_1 \leqslant 1$, $j = 1, 2$. Then clearly $\|F_j\|_1 = 1$ for each j, and hence $\|F_1\|_1 + \|F_2\|_1 = 2 = \|2F\|_1 = \|F_1 + F_2\|_1$. Now this can happen if, and only if, $\|F_1(e^{ix}) + F_2(e^{ix})\| = \|F_1(e^{ix})\| + \|F_2(e^{ix})\|$ a.e., and thus since $\mathscr{H}$ is strictly convex, we must have $F_2(e^{ix}) = \rho(e^{ix})F_1(e^{ix})$, where $\rho(e^{ix}) > 0$ a.e. Thus $F = ((1 + \rho)/2)F_1$, or $F_1(e^{ix}) = \tau(e^{ix})F(e^{ix})$, where $0 < \tau(e^{ix}) < 2$ a.e. If then E is an outer analytic unitary function in the direction of F, it follows from the remark preceding the statement of the theorem that E is also an outer analytic unitary function in the direction of F_1, so that $F_1 = f_1E$, for some $f_1 \in H^1$. Writing $F = fE$, where $f \in H^1$ and is outer, we thus have $f_1 = \tau f$. Since f is outer, by a result of de Leeuw and Rudin for the space H^1 [1, p. 474], there exist trigonometric polynomials p_n, such that $p_nf \to 1$ in H^1. Thus $\tau p_nf = p_nf_1 \to \tau$ in L^1, and since the p_nf_1 belong to H^1, so does τ. Since τ is real-valued, it is a constant, from which it follows readily that $F_1 = F_2 = F$.

Conversely, suppose that F is of norm one in $H^1_{\mathscr{H}}$ but is not outer. Then there exists a nonconstant scalar inner function q such that $\bar{q}F = G \in H^1_{\mathscr{H}}$. Let E be an outer analytic unitary function in the direction of G. Then $G = fE$ for some $f \in H^1$, and hence $F = qfE$. Of course, $\| f \|_1 = 1$. Since qf is not an outer scalar function, there exists a nonzero element $g \in H^1$ such that $\| qf \pm g \|_1 = \| qf \|_1 = 1$ [1, p. 472]. Thus gE is a nonzero element of $H^1_{\mathscr{H}}$ with $\| F \pm gE \|_1 = 1$, so that F is not an extreme point of the unit ball in $H^1_{\mathscr{H}}$.

As was mentioned earlier, the factorization of $H^1_{\mathscr{H}}$ elements into an outer analytic unitary part and a scalar H^1 function, which was used in the proof of Theorem 2, makes the following two theorems immediate corollaries of their scalar counterparts ([1, p. 470], and [6, p. 138], respectively).

THEOREM 3. *Let F be an element of the unit ball in $H^1_{\mathscr{H}}$.*

(a) *If $\| F \|_1 = 1$ and F is not an extreme point of the unit ball, then $F = \frac{1}{2}(F_1 + F_2)$, where F_1 and F_2 are extreme points of the unit ball.*

(b) *If $\| F \|_1 < 1$, then F is a convex combination of two extreme points of the unit ball.*

THEOREM 4. *An element $F \in H^\infty_{\mathscr{H}}$ is an extreme point of the unit ball in $H^\infty_{\mathscr{H}}$ if, and only if, $\| F \|_\infty \leqslant 1$ and $\int \log[1 - \| F(e^{ix}) \|] \, d\sigma = -\infty$.*

However, when we consider the closure of the set of extreme points of the unit ball in $H^1_{\mathscr{H}}$, things change significantly in the vector case. In Theorem 3 of [1, p. 470], it is shown that the closure of the set of extreme points of the unit ball in H^1 consists of all $f \in H^1$ such that $\| f \|_1 = 1$, and $f(z)$ has no zeros in the open unit disc. (For any $f \in H^1$, we use the same symbol f to denote the canonical extension of this element to a function defined on the disc.) In the vector case, we get the entire surface of the unit ball.

DEFINITION 4. A nonzero element $F \in H^1_{\mathscr{H}}$ will be said to *have constant direction* if $F(e^{ix}) = f(e^{ix})\, e$, where $f \in H^1$ and $e \in \mathscr{H}$ (i.e., if any outer analytic unitary function in the direction of F is constant).

THEOREM 5. *If $\mathscr{H}$ has dimension greater than or equal to 2, then the norm closure of the set of extreme points of the unit ball of $H^1_{\mathscr{H}}$ consists of all $F \in H^1_{\mathscr{H}}$ with $\| F \|_1 = 1$.*

Proof. Obviously, anything in the closure of this set must have norm one. If $\| F \|_1 = 1$, and F is outer, there is nothing left to prove. Thus we shall assume throughout the proof that F is an element of $H^1_{\mathscr{H}}$ with $\| F \|_1 = 1$, and F is not outer.

First suppose that F has constant direction. Then $F(e^{ix}) = f(e^{ix})\, e$, where $e \in \mathscr{H}$ with $\| e \| = 1$, and $f \in H^1$ with $\| f \|_1 = 1$. Choose $\varphi \in \mathscr{H}$ with $\| \varphi \| = 1$

and $\langle e, \varphi \rangle = 0$. Let $\{\alpha_n\}$ be an increasing sequence of positive real numbers with $\lim_n \alpha_n = 1$. For fixed n, and $\beta \in [0, 1]$, $\| \alpha_n F + \beta\varphi \|_1$ is a continuous function of β, and thus, by the connectedness of the unit interval, for each n we can find $\beta_n \in [0, 1]$ such that $\| \alpha_n F + \beta_n \varphi \|_1 = 1$. Then, for each n, $\alpha_n F + \beta_n \varphi$ is an outer function of norm one, and hence an extreme point of the unit ball of $H^1_{\mathscr{H}}$. Moreover, since $\alpha_n \to 1$, $\beta_n \to 0$ as $n \to \infty$, so that $\alpha_n F + \beta_n \varphi$ tends to F in the norm of $H^1_{\mathscr{H}}$, and hence F belongs to the closure of the set of extreme points of the unit ball of this space.

Next, suppose that F does not have constant direction. Let E be an outer analytic unitary function in the direction of F. Then $F = fE$, where $f \in H^1$ and $\|f\|_1 = 1$. Since F is not outer, by the proof of the "only if" part of Theorem 2, f cannot be an outer scalar function, so that $f = f_0 q$, where q is a nonconstant inner function and f_0 is outer.

If q is a singular function, then by [1, Theorem 3, p. 470], there exists a sequence $\{f_n\}$ of scalar outer functions with $\|f_n\|_1 = 1$ for all n, such that $f_n \to f$ in H^1. Then $\{f_n E\}$ is a sequence of outer functions in $H^1_{\mathscr{H}}$, each having norm one, such that $f_n E \to fE = F$ in $H^1_{\mathscr{H}}$, and again F belongs to the closure of the set of extreme points.

Finally, suppose that q is not a singular function, and let $q = b \cdot s$ be the factorization of q into Blaschke part b and singular part s. Let $\{e_n : n = 1, 2, \ldots\}$ be an orthonormal basis for $\mathscr{H}$ and write $F = \sum_{n \geq 1} f_n e_n$, where $f_n(e^{ix}) = \langle F(e^{ix}), e_n \rangle$. Then the coordinate functions f_n belong to H^1 for all n, and since we have assumed that F does not have constant direction, at least two of these coordinate functions, say f_1 and f_2, are nonzero elements of H^1.

Let b_0 be the Blaschke part of f_1 (i.e., b_0 is the product of b and the Blaschke part of the first coordinate function of E), and let $\{b_n\}$ be the sequence of partial products of b_0. Then $b_n \to b_0$ in the norm of H^2 [6, p. 65], and hence at least a subsequence of the b_n converges to b a.e. on the circle. If we define $F_n \in H^1_{\mathscr{H}}$ by $F_n(e^{ix}) = f_0(e^{ix})\, b_n(e^{ix})\, s(e^{ix})\, p_1(e^{ix}) + \sum_{n \geq 2} f_n(e^{ix})\, e_n$, where p_1 is the first coordinate function of E divided by its Blaschke part, then at least a subsequence of the F_n converges to F a.e. on the circle. Thus, given $\epsilon > 0$, since $\|F_n(e^{ix})\| = \|F(e^{ix})\|$ a.e., we can, by the dominated convergence theorem, find an n such that $\|F_n - F\|_1 < \epsilon$.

Let b_2' be the Blaschke part of f_2. Now b_n has only finitely many zeros all contained in some disc$\{z : |z| \leq r\}$, where $r < 1$, and b_2' has only finitely many zeros on this disc. Thus, by moving slightly the zeros of b_n if necessary, we can find a finite Blaschke product b_0', all of whose zeros are distinct from the zeros of b_2', and such that $\| b_0' - b_n \|_\infty < \epsilon$.

Now define $F_0 \in H^1_{\mathscr{H}}$ by $F_0(e^{ix}) = f_0(e^{ix})\, b_0'(e^{ix})\, s(e^{ix})\, p_1(e^{ix})\, e_1 + \sum_{n \geq 2} f_n(e^{ix})\, e_n$. Then $\|F_0\|_1 = 1$ and $\|F_0 - F_n\|_1 < \epsilon$. Moreover, in any factorization of F_0 into an outer analytic unitary function and a scalar H^1 function, the latter function can have no Blaschke part, for such a Blaschke part would have to be a Blaschke factor of all the coordinate functions of F_0, and we have rigged things to eliminate the possibility of such a common factor. Thus, by what we have already proved, F_0 belongs to the closure of the set of extreme points of the unit ball of $H^1_{\mathscr{K}}$. Since $\|F_0 - F\|_1 \leqslant \|F_0 - F_n\|_1 + \|F_n - F\|_1 < 2\epsilon$, F also belongs to the closure of this set.

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